

# CRITICAL VALUES OF GAUSSIAN $SU(2)$ RANDOM POLYNOMIALS

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ABSTRACT. In this article, we will get the estimate of the expected distribution of critical values of Gaussian  $SU(2)$  random polynomials as the degree large enough. The result is a direct application of the Kac-Rice formula.

## 1. INTRODUCTION

Random polynomials and random holomorphic functions are studied as ways to gain insight for problems arising in string theory and analytic number theory [4, 9, 14]. In [12], Kac studied and determined a formula for the expected distribution of zeros of some real Gaussian random polynomials. His work was generalized to complex random polynomials and random analytic functions throughout the years, we refer to [2, 3, 5, 11, 15] for more backgrounds and results.

**1.1.  $SU(2)$  polynomials.** When the random polynomial is defined invariant with respect to some group action, the problem can turn out to be particularly interesting, we refer §2.3 in [11] for examples. In this article, we will study a special family: the Gaussian  $SU(2)$  random polynomial. This is of particular interest in the physics literature as the zeros describe a random spin state for the Majorana representation (modulo phase) on the unit sphere [9].

Given a probability space  $\Omega$  and  $\{a_j\}_{j=1}^{\infty}$  a collection of i.i.d complex Gaussian random variables with mean 0 and variance 1 on it, the family of  $SU(2)$  random polynomials is defined as,

$$(1) \quad p_n(z) = \sum_{j=0}^n a_j \sqrt{\binom{n}{j}} z^j.$$

Although this polynomial is defined on  $\mathbb{C}$ , we may also view it as an analytic function on  $\mathbb{CP}^1 = \mathbb{C} \cup \infty$  with a pole at  $\infty$ .

Various properties of the zeros of random  $SU(2)$  polynomials have been studied such as the distribution of zeros and the two points correlation function [2, 9]. First, zeros of this polynomials are uniformly distributed on  $S^2 \cong \mathbb{CP}^1$  with respect to the Fubini-Study metric, i.e., the average distribution of zeros is invariant under the  $SU(2)$  action on  $\mathbb{CP}^1$  [11]. To be more precise, let's denote

$$\mathcal{Z}_{p_n} = \sum_{z \in \mathbb{CP}^1: p_n(z)=0} \delta_z$$

as the empirical measure of zeros of Gaussian  $SU(2)$  random polynomials and define the pairing,

$$\langle \mathcal{Z}_{p_n}, \phi \rangle = \sum_{z \in \mathbb{CP}^1: p_n(z)=0} \phi(z), \quad \text{where } \phi \in C^\infty(\mathbb{CP}^1)$$

We define the expectation,

$$\langle \mathbb{E} \mathcal{Z}_{p_n}, \phi \rangle := \mathbb{E} \langle \mathcal{Z}_{p_n}, \phi \rangle = \frac{1}{\pi^{n+1}} \int_{\mathbb{C}^{n+1}} \left( \sum_{z \in \mathbb{CP}^1: p_n(z)=0} \phi(z) \right) e^{-\frac{|a|^2}{2}} d\ell_{a_0} \cdots d\ell_{a_n}$$

where  $d\ell_{a_j} = \frac{1}{2i} da_j \wedge d\bar{a}_j$  is the Lebesgue measure on  $\mathbb{C}$ .

Then the expected density of zeros is [2],

$$\mathbb{E} \mathcal{Z}_{p_n} = n\omega_{FS},$$

in the sense that,

$$\mathbb{E} \langle \mathcal{Z}_{p_n}, \phi \rangle = n \int_{\mathbb{CP}^1} \phi \omega_{FS}, \quad \text{where } \phi \in C^\infty(\mathbb{CP}^1)$$

where  $\omega_{FS}$  is the Fubini-Study metric on  $\mathbb{CP}^1$  [8].

We can also study the two points correlation function of zeros of  $SU(2)$  polynomials and its scaling property. We define the two points correlation function as [2],

$$K_n(z, w) := \mathbb{E} (Z_{p_n}(z) \otimes Z_{p_n}(w))$$

such that for any smooth test function  $\phi_1(z) \otimes \phi_2(w)$ , we have the pairing,

$$\langle K_n(z, w), \phi_1(z) \otimes \phi_2(w) \rangle = \mathbb{E} (\langle \mathcal{Z}_{p_n}, \phi_1 \rangle) (\langle \mathcal{Z}_{p_n}, \phi_2 \rangle)$$

If we scale the two points correlation function by a factor  $\frac{1}{\sqrt{n}}$ , then we have,

$$K_n\left(\frac{z}{\sqrt{n}}, \frac{w}{\sqrt{n}}\right) = \frac{(\sinh t^2 + t^2) \cosh t - 2t \sinh t}{\sinh t^3} + O\left(\frac{1}{\sqrt{n}}\right)$$

where  $t = \frac{|z-w|^2}{2}$  and  $|z-w|$  is the geodesic distance of  $z$  and  $w$  on  $\mathbb{CP}^1$ . It's easy to see,

$$K_n\left(\frac{z}{\sqrt{n}}, \frac{w}{\sqrt{n}}\right) = t - \frac{2}{9}t^3 + O(t^5) \quad \text{as } t \rightarrow 0$$

which implies zeros repel each other. We refer to [2, 9] for more details.

**1.2. Main results.** In this article, we will study the expected distribution of nonvanishing critical values of  $|p_n|$  as  $n$  large enough.

Note that the modulus  $|p_n|$  is a subharmonic function, thus there is no local maxima; local minima are all zeros and thus nonvanishing critical values are obtained only at saddle points [6]. Hence the expected density of nonvanishing critical values of  $|p_n|$  we study in this article is in fact the expected density of values of saddle points of  $|p_n|$ .

The nonvanishing critical values of  $|p_n|$  are obtained at points,

$$(2) \quad \{z \in \mathbb{C} : p'_n = 0 \text{ and } p_n \neq 0\}.$$

For a random polynomial  $p_n$ , it has no repeated zeros almost surely, which implies that the set (2) is almost surely equivalent to,

$$(3) \quad \{z \in \mathbb{C} : p'_n = 0\}.$$

i.e., (nonvanishing)  $|p_n|$  and  $p_n$  have the same critical points almost surely.

Hence, we will first get the expected density of critical values of  $p_n$  in Theorem 1, as a direct consequence, we can apply the polar coordinate to get the expected density of nonvanishing critical values of  $|p_n|$  in Theorem 2.

We denote the empirical measure of critical values of  $p_n$  as,

$$(4) \quad \mathcal{C}_{p_n} = \sum_{z: p'_n(z)=0} \delta_{p_n(z)}.$$

We now define the pairing,

$$(5) \quad \langle \mathcal{C}_{p_n}, \phi \rangle = \sum_{z: p'_n(z)=0} \phi(p_n(z)), \quad \forall \phi(x) \in C_c^\infty(\mathbb{R}^2)$$

where  $C_c^\infty(\mathbb{R}^2)$  is the space of smooth functions on  $\mathbb{R}^2$  with compact support.

We denote  $\mathbb{D}_{p_n}(x)$  as the expected density of critical values of  $p_n$  in the sense that,

$$(6) \quad \mathbb{E}\langle \mathcal{C}_{p_n}, \phi \rangle = \int_{\mathbb{C}} \phi(x) \mathbb{D}_{p_n}(x) d\ell_x, \quad \forall \phi(x) \in C_c^\infty(\mathbb{R}^2)$$

whereas  $d\ell_x$  is the Lebesgue measure of  $\mathbb{C}$ .

Those definitions also apply to the empirical measure of the nonvanishing critical values of  $|p_n|$ ,

$$(7) \quad \mathcal{C}_{|p_n|} = \sum_{z: p'_n(z)=0} \delta_{|p_n|}.$$

which is a measure defined on the nonnegative real line  $\mathbb{R}_+$ .

We define its expectation as,

$$(8) \quad \langle \mathbb{E}\mathcal{C}_{|p_n|}, \phi \rangle := \mathbb{E}\langle \mathcal{C}_{|p_n|}, \phi \rangle = \int_0^\infty \phi(x) \mathbb{D}_{|p_n|} dx, \quad \forall \phi(x) \in C_c^\infty(\mathbb{R}_+)$$

where  $dx$  is the Lebesgue measure on  $\mathbb{R}$ .

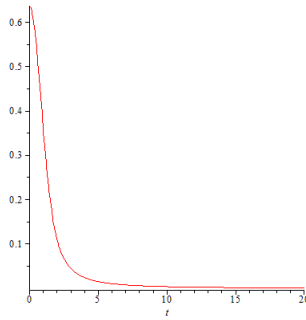
In this article, we will first get the exact formula for the expected density  $\mathbb{D}_{p_n}$  in the Proposition 1 by the Kac-Rice formula (see section §2), then we study the asymptotic behavior of  $\mathbb{D}_{p_n}$  as  $n \rightarrow \infty$ . Our main results are,

**Theorem 1.** *The expected density  $\mathbb{D}_{p_n}$  of the empirical measure  $\mathcal{C}_{p_n}$  of the critical values of  $p_n$  satisfies the estimate,*

$$(9) \quad \mathbb{D}_{p_n} = \frac{1 - e^{-|x|^2}}{\pi|x|^2} + \frac{1}{\pi} \int_0^1 e^{-(s-s \log s)|x|^2} ds + o(1) \quad \text{as } n \rightarrow \infty$$

for  $x \in \mathbb{C}$ .

It seems that the growth  $\mathbb{D}_{p_n}$  is  $\frac{1}{|x|^2}$  as  $|x|$  is away from 0 as  $n \rightarrow \infty$ , below is the computer graphic of this function with the variable  $t := |x|$ ,



As proved in Proposition 1, the density  $\mathbb{D}_{p_n} d\ell_x$  only depends on  $|x|$ , i.e., the modulus of  $|p_n|$ , thus we can rewrite it as  $\mathbb{D}_{p_n}(|x|)|x|d|x|d\theta$  under the polar coordinate. If we integrate on  $\theta$  variable, then

$$\mathbb{D}_{|p_n|} = \int_0^{2\pi} \mathbb{D}_{p_n}(|x|)|x|d\theta = 2\pi|x|\mathbb{D}_{p_n}(|x|)$$

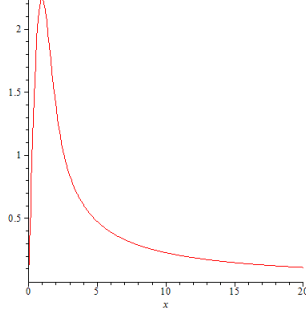
will be the density of critical values of  $|p_n|$ . Thus as a direct consequence of Theorem 1, we get,

**Theorem 2.** *The expected density  $\mathbb{D}_{|p_n|}$  of the empirical measure  $\mathcal{C}_{|p_n|}$  of the nonvanishing critical values of  $|p_n|$  satisfies the estimate,*

$$(10) \quad \mathbb{D}_{|p_n|}(x) = \frac{2(1 - e^{-x^2})}{x} + 2x \int_0^1 e^{-(s-s \log s)x^2} ds + o(1) \quad \text{as } n \rightarrow \infty$$

for  $x \in \mathbb{R}_+$ .

The growth of  $\mathbb{D}_{|p_n|}$  is also of order  $1/x$  as  $x$  is large, but there is a peak when  $x$  is small, the following is the computer graphic,



**1.3. Further remarks.** First note that our setting is different from the one in [4]. For example, in [4], critical points of  $SU(2)$  polynomials are defined to be the points

$$\{z \in \mathbb{CP}^1 : \nabla' p_n = 0\}$$

where  $\nabla' = \frac{\partial}{\partial z} - \frac{n\bar{z}dz}{1+|z|^2}$  is the smooth Chern connection on the line bundle  $\mathcal{O}(n) \rightarrow \mathbb{CP}^1$  with respect to the Fubini-Study metric and  $p_n$  is a global holomorphic section of the line bundle  $\mathcal{O}(n) \rightarrow \mathbb{CP}^1$  [8]. By choosing such smooth Chern connection, the expected distribution of critical points are also invariant under the  $SU(2)$  action [4]. But in this article, the critical points are defined by the usual derivative

$$\{z \in \mathbb{C} : \frac{\partial p_n}{\partial z} = 0\}.$$

In fact, the derivative  $\frac{\partial}{\partial z}$  is a meromorphic flat Chern connection on  $\mathcal{O}(n) \rightarrow \mathbb{CP}^1$  with a pole at  $\infty$ . Under this setting, the expected density of critical points is not  $SU(2)$  invariant, we refer to [10] for more details.

Our second remark is as following. In [7], the authors studied the expected density of non-vanishing critical values of the pointwise norm of Gaussian random holomorphic sections of the positive holomorphic line bundle over compact Kähler manifolds. Now let's briefly explain the main result in [7] and compare it with Theorem 2. Take Gaussian  $SU(2)$  random polynomials (sections)  $p_n$  for example. We equip the line bundle  $\mathcal{O}(n) \rightarrow \mathbb{CP}^1$  with a Hermitian metric  $h^n = e^{-n\phi}$  where  $\phi = \log(1 + |z|^2)$  is the Kähler potential of Fubini-Study metric. Then the pointwise  $h$ -norm of the holomorphic section  $|p_n|_{h^n} = |p_n|e^{-\frac{n\phi}{2}}$  is global defined on  $\mathbb{CP}^1$  [8] and hence the critical points of  $|p_n|_{h^n}$  is defined as,

$$\Sigma_n = \{z \in \mathbb{CP}^1 : \frac{\partial |p_n|_{h^n}}{\partial z} = 0\}$$

We define the (normalized) empirical measure of critical value of  $|p_n|_{h^n}$  as,

$$\mathcal{C}_{|p_n|_{h^n}} := \frac{1}{n} \left( \sum_{z \in \Sigma_n} \delta_{|p_n|_{h^n}} \right)$$

which is also a measure defined on  $\mathbb{R}_+$ .

Then the expectation of  $\mathcal{C}_{|p_n|_{h^n}}$  satisfies the estimate,

$$(11) \quad \mathbb{E}\mathcal{C}_{|p_n|_{h^n}} = x \left( 2x^2 - 4 + 8e^{-\frac{x^2}{2}} \right) e^{-x^2} + O\left(\frac{1}{n}\right), \quad x \in \mathbb{R}_+$$

as  $n$  large enough. In fact, this estimate is universal: it holds on any Riemannian surfaces [7].

Thus the (normalized) density  $\mathbb{E}\mathcal{C}_{|p_n|_{h^n}}$  is exponent decaying as  $x$  large enough which is quite different from the behavior of (non-normalized) density  $\mathbb{E}\mathcal{C}_{|p_n|}$  in Theorem 2. This is mainly because of the connection we choose: the usual derivative  $\frac{\partial}{dz}$  in this article is a meromorphic flat connection on  $\mathbb{CP}^1$  with a pole at  $\infty$  while in [7], the proof of (11) relies on a choice of smooth Chern connection  $\nabla' = \frac{\partial}{\partial z} - \frac{n\bar{z}dz}{1+|z|^2}$ .

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## 2. KAC-RICE FORMULA

In this section, we first review the Kac-Rice formula for a stochastic process, referring to [1, 12, 13] for more details. Then we generalize the formula to the expected distribution of critical values of  $p_n$ .

The Kac-Rice formula is as follows: let  $f(z)$  be a real valued stochastic process indexed by a compact interval  $I \subset \mathbb{R}$ . Then the Kac-Rice formula for the expected number of zeros is,

$$\mathbb{E}\#\{z \in I : f(z) = 0\} = \int_I \int_{\mathbb{R}} |y| p_z(0, y) dy dz$$

where  $p_z(0, y)$  is the joint density  $p_z(x, y)$  of  $(f, f')$  evaluated at  $(0, y)$ . If  $f$  is a Gaussian process, then the joint density  $p_z(x, y)$  is determined by the covariance matrix of  $(f, f')$  [1].

The proof of this formula is explained in more details in [1]. The idea of the proof is based on the following observation,

$$\#\{z \in I : f(z) = 0\} = \int_I \delta_0(f(z)) |f'(z)| dz$$

We take expectation on both sides to get,

$$\begin{aligned} \mathbb{E}\#\{z \in I : f(z) = 0\} &= \int_I \int_{\mathbb{R}_y} \int_{\mathbb{R}_x} \delta_0(x) p_z(x, y) |y| dx dy dz \\ &= \int_I \int_{\mathbb{R}} |y| p_z(0, y) dy dz \end{aligned}$$

Thus the expected density of zeros of  $f$  is given by,

$$(12) \quad \mathbb{E} \left( \sum_{z \in I: f(z)=0} \delta_z \right) = \left( \int_{\mathbb{R}} |y| p_z(0, y) dy \right) dz$$

If  $f(z)$  is a complex stochastic process indexed by a compact complex domain, then the above formula reads,

$$(13) \quad \mathbb{E} \left( \sum_{z \in I: f(z)=0} \delta_z \right) = \left( \int_{\mathbb{C}} |y|^2 p_z(0, y) d\ell_y \right) d\ell_z$$

where  $d\ell_y$  and  $d\ell_z$  are Lebesgue measures on  $\mathbb{C}$ . Compared with (12), we get  $|y|^2$  since a 1-dimensional complex random process is a 2-dimensional real random process. In fact, this formula is based on the definition of the delta function,

$$\#\{z \in I : f(z) = 0\} = \int_I \delta_0(f(z)) \frac{1}{2i} df \wedge d\bar{f} = \int_I \delta_0(f(z)) |f'|^2 d\ell_z$$

The formula arises when we take expectation on both sides.

**2.1. Kac-Rice formula: Revisit.** In this subsection, let's get the formula for the expected density of critical values of a (real or complex) stochastic process  $f$  by the method of Kac-Rice.

For simplicity, let's first consider a smooth real stochastic process  $f \in C^\infty(I)$  where  $I$  is a compact subset in  $\mathbb{R}$ .

Let  $\Theta \subset \mathbb{R}$  be a compact subset. Let's denote the set of critical values in  $\Theta$  as,

$$\mathcal{C}_\Theta = \{z \in I : f(z) \in \Theta, f'(z) = 0\}.$$

Let's denote the measure  $\mu(x)dx$  on  $\Theta$  as,

$$\mu(x)dx = \mathbb{E} \left( \sum_{z \in \mathcal{C}_\Theta} \delta_{f(z)} \right)$$

in the sense that,

$$\mathbb{E} \left( \left\langle \sum_{z \in \mathcal{C}_\Theta} \delta_{f(z)}, \phi \right\rangle \right) = \int_\Theta \phi \mu(x) dx$$

where  $\phi$  is any smooth test function defined on  $\Theta$ .

Then we have the following lemma,

**Lemma 1.** *Let's denote  $p_z(x, y, \xi)$  as the joint probability of  $(f, f', f'')$ . Then,*

$$\mu(x)dx = \left( \int_I \int_{\mathbb{R}} |\xi| p_z(x, 0, \xi) d\xi dz \right) dx$$

where  $dx$ ,  $d\xi$  and  $dz$  are Lebesgue measures on  $\mathbb{R}$ .

*Proof.* First note that,

$$\left\langle \sum_{f \in \Theta, f'=0} \delta_{f(z)}, \phi(x) \right\rangle = \sum_{f \in \Theta, f'=0} \phi(f(z)) = \int_I \chi_{\{f \in \Theta\}} \phi(f(z)) \delta(f') df'$$

By taking expectation on both sides,

$$\begin{aligned} & \mathbb{E} \left\langle \sum_{f \in \Theta, f'=0} \delta_{f(z)}, \phi(x) \right\rangle \\ &= \int_{\mathbb{R}_x} \int_I \int_{\mathbb{R}_\xi} \int_{\mathbb{R}_y} p_z(x, y, \xi) \chi_{\{x \in \Theta\}} \phi(x) \delta(y) |\xi| dy d\xi dz dx \\ &= \int_\Theta \left( \int_I \int_{\mathbb{R}_\xi} p_z(x, 0, \xi) |\xi| d\xi dz \right) \phi(x) dx \\ &= \int_\Theta \phi(x) \mu(x) dx \end{aligned}$$

□

In the proof of Lemma 1, we have assumed  $I$  and  $\Theta$  being compact subsets in  $\mathbb{R}$ . But the proof of Lemma 1 can be generalized to the  $SU(2)$  random polynomials  $p_n$  which are a collection of complex Gaussian stochastic processes indexed by  $\mathbb{C}$ .

The generalization of  $\Theta$  to be  $\mathbb{C}$  only requires picking up a sequence of discs centered at the origin with radius  $m \in \{1, 2, \dots\}$  and taking limit in weak sense. And the generalization from  $I$  to  $\mathbb{C}$  is the same.

However, we do need to modify the pairing by choosing the test functions  $\phi(z)$  in the smooth compact supported space  $\mathcal{C}_c^\infty(\mathbb{R}^2)$  in order to change the order of the integration on  $\mathbb{C}$ . Following the proof of Lemma 1, we have,

**Lemma 2.** *The expected density of critical values of  $p_n$  is,*

$$(14) \quad \mathbb{D}_{p_n} d\ell_x = \left( \int_{\mathbb{C}} \int_{\mathbb{C}} |\xi|^2 p_z(x, 0, \xi) d\ell_\xi d\ell_z \right) d\ell_x$$

where  $d\ell_x, d\ell_\xi$  and  $d\ell_z$  are Lebesgue measures on  $\mathbb{C}$  and

$$(15) \quad p_z(x, 0, \xi) = \frac{1}{\pi^3 \det \Delta_z} \exp \left\{ - \left\langle \begin{pmatrix} x \\ 0 \\ \xi \end{pmatrix}, \Delta_z^{-1} \begin{pmatrix} \bar{x} \\ 0 \\ \bar{\xi} \end{pmatrix} \right\rangle \right\}$$

is the joint density of  $(p_n, p'_n, p''_n)$  where  $\Delta_z$  is the covariance matrix of  $(p_n, p'_n, p''_n)$ .

The proof of this formula is the same as the one in Lemma 1. Note that we get  $|\xi|^2$  in the formula since  $p_n$  is a 1-dimensional complex Gaussian process which is a 2-dimensional real process. Moreover,  $p_n$  is a Gaussian process, hence the joint density  $p_z(x, y, \xi)$  is uniquely determined by the covariance matrix of  $(p_n, p'_n, p''_n)$ , we refer to [1, 2, 4] for more details.

### 3. PROOF OF MAIN THEOREMS

**3.1. The Density  $\mathbb{D}_{p_n}$ .** In this subsection, we will derive the exact formula for  $\mathbb{D}_{p_n}$  based on Lemma 2. We prove,

**Proposition 1.** *The expected density of the empirical measure of  $\mathcal{C}_{p_n}$  is given by the formula,*

$$(16) \quad \mathbb{D}_{p_n} = \frac{n-1}{\pi} \int_1^\infty \frac{n(r-1)+1}{r^{n+2}} e^{-\frac{n(r-1)+1}{r^n} |x|^2} dr$$

Thus  $\mathbb{D}_{p_n}$  is a function only depending on  $|x|$ .

*Proof.* By Lemma 2, in order to compute the expected density of critical values of  $p_n$ , we first need to compute the covariance matrix of  $(p_n, p'_n, p''_n)$ .

By definition, the covariance matrix of the Gaussian process  $(p_n, p'_n, p''_n)$  is given by [1, 4],

$$\Delta = \begin{pmatrix} \mathbb{E}(p_n \overline{p_n}) & \mathbb{E}(p'_n \overline{p_n}) & \mathbb{E}(p''_n \overline{p_n}) \\ \mathbb{E}(p_n \overline{p'_n}) & \mathbb{E}(p'_n \overline{p'_n}) & \mathbb{E}(p''_n \overline{p'_n}) \\ \mathbb{E}(p_n \overline{p''_n}) & \mathbb{E}(p'_n \overline{p''_n}) & \mathbb{E}(p''_n \overline{p''_n}) \end{pmatrix}$$

The covariance kernel for the Gaussian process  $p_n$  is,

$$\mathbb{E}(p_n(z) \overline{p_n(w)}) := \Pi_n(z, w) = (1 + z\bar{w})^n$$

Then we can express each entry in the covariance matrix as,

$$\mathbb{E}(p_n \overline{p_n}) = \Pi_n(z, z), \quad \mathbb{E}(p'_n \overline{p_n}) = \frac{\partial \Pi_n(z, w)}{\partial z} \Big|_{z=w}$$

$$\mathbb{E}(p''_n \overline{p_n}) = \frac{\partial^2 \Pi_n(z, w)}{\partial^2 z} \Big|_{z=w}, \quad \mathbb{E}(p'_n \overline{p'_n}) = \frac{\partial^2 \Pi_n(z, w)}{\partial z \partial \bar{w}} \Big|_{z=w}$$

$$\mathbb{E}(p_n'' \overline{p_n}) = \frac{\partial^3 \Pi_n(z, w)}{\partial^2 z \partial \bar{w}} \Big|_{z=w}, \quad \mathbb{E}(p_n'' \overline{p_n''}) = \frac{\partial^4 \Pi_n(z, w)}{\partial^2 z \partial^2 \bar{w}} \Big|_{z=w}$$

Some straight forward computations show that the covariance matrix is,

$$\Delta_z = (1 + |z|^2)^n \begin{pmatrix} 1 & \frac{n\bar{z}}{1+|z|^2} & \frac{n(n-1)\bar{z}^2}{(1+|z|^2)^2} \\ \frac{nz}{1+|z|^2} & \frac{n+n^2|z|^2}{(1+|z|^2)^2} & \frac{2n(n-1)\bar{z}+(n-1)n^2\bar{z}|z|^2}{(1+|z|^2)^3} \\ \frac{n(n-1)z^2}{(1+|z|^2)^2} & \frac{2n(n-1)z+(n-1)n^2z|z|^2}{(1+|z|^2)^3} & \frac{2n(n-1)+4n(n-1)^2|z|^2+n^2(n-1)^2|z|^4}{(1+|z|^2)^4} \end{pmatrix},$$

Hence,

$$(17) \quad \det \Delta_z = (1 + |z|^2)^{3n} \frac{2n^3 - 2n^2}{(1 + |z|^2)^6}.$$

Now we denote,

$$Q_z(x, \xi) =: \left\langle \begin{pmatrix} x \\ 0 \\ \xi \end{pmatrix}, \Delta_z^{-1} \begin{pmatrix} \bar{x} \\ 0 \\ \bar{\xi} \end{pmatrix} \right\rangle$$

Then by direct computations, we rewrite,

$$Q_z(x, \xi) = \frac{(1 + |z|^2)^{2n}}{\det \Delta_z} \left\langle \begin{pmatrix} x \\ 0 \\ \xi \end{pmatrix}, \begin{pmatrix} \frac{n^5|z|^4+2n^4(|z|^2-|z|^4)+n^3(|z|^4-2|z|^2+2)-2n^2}{(1+|z|^2)^6} & \frac{(n^3-n^2)\bar{z}^2}{(1+|z|^2)^4} \\ \frac{(n^3-n^2)z^2}{(1+|z|^2)^4} & \frac{n}{(1+|z|^2)^2} \end{pmatrix} \begin{pmatrix} \bar{x} \\ 0 \\ \bar{\xi} \end{pmatrix} \right\rangle.$$

We expand this expression and further rewrite  $Q_z(x, \xi)$  as,

$$(18) \quad \frac{1}{2(1 + |z|^2)^n} \left( \left| \sqrt{n^2 - n} \bar{z}^2 x + \frac{1}{\sqrt{n^2 - n}} \xi (1 + |z|^2)^2 \right|^2 + 2(n|z|^2 + 1)|x|^2 \right).$$

By Lemma 2, the expected density of critical values of  $p_n$  is given by the formula,

$$(19) \quad \mathbb{D}_{p_n}(x) = \frac{1}{\pi^3} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{e^{-Q_z(x, \xi)}}{\det \Delta_z} |\xi|^2 d\ell_{\xi} d\ell_z.$$

Let's integrate  $\xi$  variable first. Plug (18) into (19), we can rewrite (19) as,

$$(20) \quad \mathbb{D}_{p_n}(x) = \frac{1}{\pi^3} \int_{\mathbb{C}} K_z \frac{e^{-\frac{n|z|^2+1}{(1+|z|^2)^n} |x|^2}}{\det \Delta_z} d\ell_z.$$

where  $K_z$  is the following integral in  $\xi$  variable,

$$K_z = \int_{\mathbb{C}} \exp \left\{ -\frac{1}{2(1 + |z|^2)^n} \left| \sqrt{n^2 - n} \bar{z}^2 x + \frac{1}{\sqrt{n^2 - n}} \xi (1 + |z|^2)^2 \right|^2 \right\} |\xi|^2 d\ell_{\xi}.$$

Change variables  $\xi \rightarrow \frac{1}{\sqrt{n^2 - n}} \xi (1 + |z|^2)^2$  to get,

$$K_z = \frac{(n^2 - n)^2}{(1 + |z|^2)^8} \int_{\mathbb{C}} \exp \left\{ -\frac{1}{2(1 + |z|^2)^n} \left| \sqrt{n^2 - n} x \bar{z}^2 + \xi \right|^2 \right\} |\xi|^2 d\ell_{\xi}.$$

Further change variable,  $\xi \rightarrow \sqrt{n^2 - n} x \bar{z}^2 + \xi$  to get,

$$K_z = \frac{(n^2 - n)^2}{(1 + |z|^2)^8} \int_{\mathbb{C}} \exp \left\{ -\frac{|\xi|^2}{2(1 + |z|^2)^n} \right\} \left| \xi - \sqrt{n^2 - n} x \bar{z}^2 \right|^2 d\ell_{\xi},$$

which equals to

$$K_z = \frac{(n^2 - n)^2}{(1 + |z|^2)^8} \int_{\mathbb{C}} \exp \left\{ -\frac{|\xi|^2}{2(1 + |z|^2)^n} \right\} (|\xi|^2 + (n^2 - n)|x \bar{z}^2|^2) d\ell_{\xi}.$$



Integrate it out, we have,

$$K_z = \pi \frac{(n^2 - n)^2}{(1 + |z|^2)^8} [2(1 + |z|^2)^n (n^2 - n) |x|^2 |z|^4 + 4(1 + |z|^2)^{2n}].$$

Now we change variable  $r = 1 + |z|^2$ , rewrite

$$K_z = \pi \frac{(n^2 - n)^2}{r^8} [2r^n (n^2 - n) |x|^2 (r - 1)^2 + 4r^{2n}]$$

and

$$\det \Delta_z = r^{3n-6} (2n^3 - 2n^2), \quad e^{-\frac{n|z|^2+1}{(1+|z|^2)^n} |x|^2} = e^{-\frac{n(r-1)+1}{r^n} |x|^2}$$

Plug these two lines back into the formula of (20) and use the polar coordinate  $d\ell_z = \frac{1}{2} dr d\theta$ , integrate on  $\theta$  variable, we can rewrite  $\mathbb{D}_{p_n}$  as,

$$(21) \quad \mathbb{D}_{p_n} = \frac{n-1}{\pi} \int_1^\infty \frac{(n^2 - n) r^n (r-1)^2 |x|^2 + 2r^{2n}}{r^{3n+2}} e^{-\frac{n(r-1)+1}{r^n} |x|^2} dr$$

There are two parts in the numerator, we use integration by part to simplify the first part. Note that

$$de^{-\frac{n(r-1)+1}{r^n} |x|^2} = e^{-\frac{n(r-1)+1}{r^n} |x|^2} [r^{-n-1} (n^2 - n) (r-1) |x|^2] dr,$$

then the first part is equal to,

$$\begin{aligned} & \frac{n-1}{\pi} \int_1^\infty \frac{(n^2 - n) r^n (r-1)^2 |x|^2}{r^{3n+2}} e^{-\frac{n(r-1)+1}{r^n} |x|^2} dr \\ &= \frac{n-1}{\pi} \int_1^\infty \frac{(r-1)}{r^{n+1}} de^{-\frac{n(r-1)+1}{r^n} |x|^2} \\ &= \frac{n-1}{\pi} \int_1^\infty \left[ \frac{n}{r^{n+1}} - \frac{n+1}{r^{n+2}} \right] e^{-\frac{n(r-1)+1}{r^n} |x|^2} dr \end{aligned}$$

Hence the density (21) is further simplified to be,

$$\mathbb{D}_{p_n}(|x|^2) = \frac{n-1}{\pi} \int_1^\infty \frac{n(r-1)+1}{r^{n+2}} e^{-\frac{n(r-1)+1}{r^n} |x|^2} dr$$

which completes the proof.  $\square$

### 3.2. Proof of Theorem 1.

Now we turn to the proof of our main Theorem 1. We denote  $t = \frac{1}{r}$  and

$$y_n(t) = \frac{n(r-1)+1}{r^n} = nt^{n-1} - (n-1)t^n,$$

then we have  $t \in [0, 1]$  and  $y_n(t) \in [0, 1]$  with  $y_n(0) = 0$  and  $y_n(1) = 1$ .

Substitute  $\frac{n(r-1)+1}{r^n}$  by  $y_n(t)$ , we rewrite  $\mathbb{D}_{p_n}$  in Proposition (1) as,

$$\begin{aligned} \mathbb{D}_{p_n} &= \frac{n-1}{\pi} \int_1^\infty \frac{y_n(t)}{r^2} e^{-y_n(t) |x|^2} dr \\ &= \frac{n-1}{\pi} \int_0^1 y_n(t) e^{-y_n(t) |x|^2} dt \end{aligned}$$

where in the last step, we change variable  $t \rightarrow \frac{1}{r}$ .

The trick to estimate  $\mathbb{D}_{p_n}$  is to calculate

$$g_n(|x|^2) := \int_0^1 e^{-y_n(t) |x|^2} dt,$$

By integration by part, we have,

$$\begin{aligned}
g_n(|x|^2) &= \int_0^1 t' e^{-y_n(t)|x|^2} dt \\
&= e^{-|x|^2} + \int_0^1 t y_n'(t) |x|^2 e^{-y_n(t)|x|^2} dt \\
&= e^{-|x|^2} + n(n-1)|x|^2 \int_0^1 (t^{n-1} - t^n) e^{-y_n(t)|x|^2} dt \\
&= e^{-|x|^2} + n|x|^2 \int_0^1 (nt^{n-1} - (n-1)t^n) e^{-y_n(t)|x|^2} dt - n|x|^2 \int_0^1 t^{n-1} e^{-y_n(t)|x|^2} dt \\
&= e^{-|x|^2} + n|x|^2 \int_0^1 y_n(t) e^{-y_n(t)|x|^2} dt - |x|^2 h_n(|x|^2) \\
&= e^{-|x|^2} + \frac{\pi n |x|^2}{n-1} \mathbb{D}_{p_n} - |x|^2 h_n(|x|^2),
\end{aligned}$$

where we denote

$$(22) \quad h_n(|x|^2) := n \int_0^1 t^{n-1} e^{-y_n(t)|x|^2} dt.$$

Thus,

$$(23) \quad \mathbb{D}_{p_n} = \frac{n-1}{n\pi} \left( \frac{g_n(|x|^2) - e^{-|x|^2}}{|x|^2} + h_n(|x|^2) \right)$$

We claim

$$\lim_{n \rightarrow \infty} g_n(|x|^2) = 1.$$

This is quite straight forward, as  $\forall \epsilon \in (0, 1)$ , we rewrite,

$$g_n(|x|^2) = \int_0^1 e^{-y_n(t)|x|^2} dt = \int_0^{1-\epsilon} + \int_{1-\epsilon}^1$$

Since  $y_n(t) \rightarrow 0$  uniformly on  $[0, 1-\epsilon]$  as  $n \rightarrow \infty$ , thus

$$\lim_{n \rightarrow \infty} \int_0^{1-\epsilon} e^{-y_n(t)|x|^2} dt = \int_0^{1-\epsilon} \lim_{n \rightarrow \infty} e^{-y_n(t)|x|^2} dt = 1 - \epsilon$$

For the second integration, since  $y_n(t) \geq 0$  on  $[0, 1]$ , we have  $\int_{1-\epsilon}^1 e^{-y_n(t)|x|^2} dt \leq \epsilon$  Hence we get

$$1 - \epsilon \leq \lim_{n \rightarrow \infty} \int_0^1 e^{-y_n(t)|x|^2} dt \leq \overline{\lim}_{n \rightarrow \infty} \int_0^1 e^{-y_n(t)|x|^2} dt \leq 1$$

As  $\epsilon$  is chosen arbitrarily, letting  $\epsilon \rightarrow 0^+$  yields the claim.

Now we estimate (23) to be,

$$\begin{aligned}
(24) \quad \mathbb{D}_{p_n} &= \frac{n-1}{n\pi} \left( \frac{1 - e^{-|x|^2}}{|x|^2} + h_n(|x|^2) + o(1) \right) \\
&= \frac{1 - e^{-|x|^2}}{\pi |x|^2} + \frac{1}{\pi} h_n(|x|^2) + o(1).
\end{aligned}$$

as  $n \rightarrow \infty$ .

We now turn to estimate  $h_n(|x|^2)$ . Change variable  $s = t^n$ ,  $h_n$  will be rewritten as

$$\int_0^1 e^{z_n(s)|x|^2} ds$$

where

$$z_n(s) = -ns^{\frac{n-1}{n}} + (n-1)s.$$

It's easy to check that

$$z_n(s) \leq z_{n+1}(s)$$

for any fixed  $s \in [0, 1]$ .

Thus we have  $z_n(s)$  monotone increasing to  $-(s - s \log s)$  as  $n \rightarrow \infty$ , hence,  $h_n(|x|^2)$  will satisfy

$$\lim_{n \rightarrow \infty} h_n(|x|^2) = \int_0^1 e^{-(s-s \log s)|x|^2} ds$$

This will give us the estimate

$$h_n(|x|^2) = \int_0^1 e^{-(s-s \log s)|x|^2} ds + o(1)$$

as  $n \rightarrow \infty$ .

Hence we further estimate (24) to be,

$$\mathbb{D}_{p_n} = \frac{1 - e^{-|x|^2}}{\pi|x|^2} + \frac{1}{\pi} \int_0^1 e^{-(s-s \log s)|x|^2} ds + o(1) \text{ as } n \rightarrow \infty$$

which completes the proof of Theorem 1.

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